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A Representation Theory for Solutions of a Higher Order Heat Equation, I

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1. INTRODUCTION

The higher order heat equation given by

$$(-1)^{q+1} \frac{\partial^{2q}}{\partial x^{2q}} u(x, t) = \frac{\partial}{\partial t} u(x, t), \tag{1.1}$$

q a positive integer, was studied by J. B. Diaz and C. S. Means [2]. They dealt with existence and uniqueness theorems for the initial value problem. In the classical case, $q=1$, P. C. Rosenbloom and D. V. Widder [10] derived series representation theories for expanding solutions of the heat equation in terms of polynomial solutions and associated functions. These results were extended to the generalized heat equation $u_{xx} + (2\gamma/x)u_x = u_t$ by D. T. Haimo [7] and L. R. Bragg [1], to the equation $u_{xx} + (2\gamma/x + (2\gamma/x + B'(x)/B(x))u_x = u_t$, B a suitable analytic function, by A. Fitouhi [5], and to the Laguerre differential heat equation $xu_{xx} + (\alpha + 1 - x)u_x = u_t$ by

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F. M. Cholewinski and D. T. Haimo [8]. H. Kemnitz, in [9], dealt with a variation of Eq. (1.1).

The extension of the classical results to the higher order case presents some special problems. In particular, the source solution $G_1(x; t)$ that plays such a central role in the theory of the ordinary heat equation is defined by an integral which represents an elementary exponential function and is readily seen to be positive for all values of the space variable. Moreover, the associated functions are related to the heat polynomials by a simple Appell transformation, $u(x, t) \to G_1(x, t)\, u(x/t, -1/t)$.

In the case of Eq. (1.1) with $q > 1$, however, the source solution is represented by hypergeometric functions and it is known to assume both positive and negative values in any interval of its space variable [2]. Further, a direct analogue of the Appell transformation fails to exist. Taking into account difficulties arising from these factors, in this paper, we obtain a useful estimate for the source solution and its derivatives, define a set of polynomial solutions and their associates, and derive a variety of their properties, including their relationship to some hypergeometric series and to generalized Hermite polynomials and their associates. In its sequel, II, we establish the biorthogonality of our fundamental sets, and obtain sufficient conditions for the representation of a solution of (1.1) in a series of polynomial solutions with coefficients involving the associated functions.

2. Source Solution

We consider the higher-order heat equation

$$(-1)^{q+1} \frac{\partial^{2q}}{\partial x^{2q}} u(x, t) = \frac{\partial}{\partial t} u(x, t), \tag{2.1}$$

and denote by H the class of all function $u(x, t)$ which belong to C^{2q} in x and to C^1 in t and which satisfy Eq. (2.1). Such functions u will be called temperatures in line with the classical case $q = 1$.

A fundamental role is assumed by the temperature $G_q(x; y)$, the source solution of (2.1), defined for any x, and for $t > 0$, by the integral

$$\begin{aligned} G_q(s, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-tu^{2q} + ixu}\, du \\ &= \frac{1}{\pi} \int_0^{\infty} e^{-tu^{2q}} \cos xu\, du. \end{aligned} \tag{2.2}$$

With a change of variable, we have

$$G_q(x;t)=\frac{1}{2q\pi}t^{-1/(2q)}\int_0^\infty e^{-u}\cos\left(x\left(\frac{u}{t}\right)^{1/2q}\right)u^{(1-2q)/2q}\,du$$

$$=\frac{1}{2q\pi}t^{-1/(2q)}\int_0^\infty e^{-u}u^{(1-2q)/2q}\,du\sum_{k=0}^{\infty}(-1)^k\frac{(x(u/t)^{1/2q})^{2k}}{(2k)!}$$

$$=\frac{1}{2q\pi}t^{-1/(2q)}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!\,t^{k/q}}\int_0^\infty e^{-u}u^{(1+2k)/2q-1}\,du,$$

so that

$$G_q(x;t)=\frac{1}{2q\pi}t^{-1/(2q)}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma((2k+1)/2q)}{\Gamma(2k+1)}\left(\frac{x^2}{t^{1/q}}\right)^k, \tag{2.3}$$

where termwise integration follows by Fubini's theorem.

We note that

$$G_1(x;t)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t},$$

the simple classical source solution. When $q=2$,

$$G_2(x;t)=\frac{1}{2^{3/2}t^{1/4}}\sum_{k=0}^{\infty}\frac{(-x^2/8t^{1/2})^k}{k!\,\Gamma((2k+3)/4)}$$

$$=\frac{1}{2^{3/2}t^{1/4}}\phi\left(\frac{1}{2},\frac{3}{4};-\frac{x^2}{8t^{1/2}}\right),$$

where $\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}(z^k/k!\,\Gamma(\alpha k+\beta))$, see [2].

The source solution can also be expanded in a series of hypergeometric functions, since we have, for $k=qm+l$, $l=0,1,\ldots,q-1$, $m=0,1,\ldots,$

$$G_q(x;t)=\frac{1}{2q\pi}t^{-1/(2q)}\sum_{l=0}^{q-1}(-1)^l\left(\frac{x}{1/t^{2q}}\right)^{2l}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^{qm}\Gamma\left(\dfrac{2qm+2l+1}{2q}\right)}{\Gamma(2qm+2l+1)}\left(\frac{x}{1/t^{2q}}\right)^{2qm}$$

$$=\frac{1}{2q\pi}t^{-1/(2q)}\sum_{l=0}^{q-1}\frac{(-1)^l\Gamma\left(\dfrac{2l+1}{2q}\right)}{\Gamma(2l+1)}\left(\frac{x}{1/t^{2q}}\right)^{2l}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^{qm}\left(\dfrac{2l+1}{2q}\right)_m}{(2l+1)_{2qm}}\left(\frac{x}{1/t^{2q}}\right)^{2qm},$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$.

Appealing to the Gauss–Legendre multiplication formula [3, p. 4], we find that

$$G_q(x, t) = \frac{1}{2q\pi} t^{-1/(2q)}$$
$$\times \sum_{l=0}^{q-1} \frac{(-1)^l \Gamma\left(\frac{2l+1}{2q}\right)}{\Gamma(2l+1)} \left(\frac{x}{t^{-1/(2q)}}\right)^{2l}$$
$$\times \sum_{m=0}^{\infty} \frac{(-1)^{qm}(xt^{-1/(2q)})^{2qm}}{(2q)^{2qm} m! \left(\frac{2l+2}{2q}\right)_m \cdots \left(\frac{2q-1}{2q}\right)_m \left(\frac{2q+1}{2q}\right)_m \cdots \left(\frac{2l+2q}{2q}\right)_m},$$

and

$$G_q(x; t) = \frac{1}{2q\pi} t^{-1/(2q)} \sum_{l=0}^{q-1} \frac{(-1)^l \Gamma\left(\frac{2l+1}{2q}\right)}{\Gamma(2l+1)} \left(\frac{x}{1/t^{2q}}\right)^{2l}$$
$$\times {}_0F_{2q-2}\left(-; \frac{2l+2}{2q}, \frac{2l+3}{2q}, \ldots, \frac{2q-1}{2q}, \frac{2q+1}{2q}, \ldots, \frac{2l+2q}{2q}; \frac{(-1)^q}{t}\left(\frac{x}{2q}\right)^{2q}\right). \tag{2.4}$$

Using the integral representation (2.2), we note that the $(2q-1)$th derivative reproduces the source solution in that, if D_x denotes differentiation with respect to x,

$$D_x^{2q-1} G_q(x; t) = \frac{1}{\pi} \int_0^{\infty} e^{-tu^{2q}} u^{2q-1} (-1)^q \sin xu \, du$$
$$= \frac{(-1)^q}{\pi} \frac{x}{2qt} \int_0^{\infty} e^{-tu^{2q}} \cos xu \, du,$$

where the last integral was derived by an integration by parts. Thus we have

$$D_x^{2q-1} G_q(x; t) = (-1)^q \frac{x}{2qt} G_q(x; t). \tag{2.5}$$

For $q > 1$, the source solution loses some of its simpler properties so that in order to exploit its role in temperature representations, suitable estimates for $G_q(x; t)$ and its derivatives become essential. In [6], A. Friedman establishes estimates for the derivatives of a source solution of a parabolic system. A modification of that approach leads us first to the following preliminary inequality.

LEMMA 2.1. *Given γ, $0 \leqslant \gamma < 1$, there is a constant $A_{q,\gamma} \geqslant 1$ such that, for all real x, y,*

$$-\operatorname{Re}(x+iy)^{2q} \leqslant -\gamma x^{2q} + A_{q,\gamma}\, y^{2q}. \tag{2.6}$$

For $q = 1, 2$, we may choose $A_{1,\gamma} = 1$, $A_{2,\gamma} = (8+\gamma)/(1-\gamma)$, $0 \leqslant \gamma < 1$.

We note that, for $t \geqslant 0$, we have

$$\begin{aligned} |e^{-t(x+iy)^{2q}}| &= e^{-t \operatorname{Re}(x+iy)^{2q}} \\ &\leqslant e^{-\gamma t x^{2q} + A_{q,\gamma} t y^{2q}}. \end{aligned} \tag{2.7}$$

Since, for $t > 0$,

$$G_q(x;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-tu^{2q} + ixu}\, du,$$

and since, by Cauchy's theorem, the integral is independent of the path of integration,

$$G_q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(u+iv)^{2q} + ix(u+iv)}\, du,$$

where v will be suitably chosen. Taking note of (2.7), we find that

$$\begin{aligned} |G_q(x;t)| &\leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t\gamma u^{2q} + tA_{q,\gamma} v^{2q} - vx}\, du \\ &\leqslant \frac{1}{\pi} e^{tA_{q,\gamma} v^{2q} - vx} \int_0^{\infty} e^{-t\gamma u^{2q}}\, du \\ &\leqslant \frac{\Gamma(1/2q)}{2q\pi} (t\gamma)^{-1/2q} e^{tA_{q,\gamma} v^{2q} - vx}, \end{aligned}$$

or, if

$$v = \frac{|x|}{x} \left(\frac{|x|}{2qtA_{q,\gamma}} \right)^{1/(2q-1)},$$

we have

$$|G_q(x,t)| \leqslant \frac{\Gamma(1/2q)}{2q\pi} (t\gamma)^{-1/2q} \exp\left[-\frac{2q-1}{2q} \left(\frac{|x|^{2q}}{2qtA_{q,\gamma}} \right)^{1/(2q-1)} \right]. \tag{2.8}$$

For the derivatives of $G_q(x; t)$, we similarly have the representation

$$D_x^m G_q(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(u+iv)^{2q} + ix(u+iv)} [i(u+iv)]^m \, du,$$

so that

$$|D_x^m G_q(x; t)| \leqslant \frac{1}{\pi} e^{tA_{q,\gamma} v^{2q} - vx} \int_0^{\infty} e^{-t\gamma u^{2q}} (u^2+v^2)^{m/2} \, du.$$

Using the elementary inequality

$$y^a \leqslant e^{cy^b} \left(\frac{a}{bec}\right)^{a/b}, \qquad y \geqslant 0, \qquad a, b, c > 0, \tag{2.9}$$

we have

$$|D_x^m G_q(x; t)| \leqslant \frac{1}{\pi} e^{tA_{q,\gamma} v^{2q} - vx} \int_0^{\infty} e^{-t\gamma u^{2q} + \varepsilon(u^2+v^2)^q} \left(\frac{m}{2qe\varepsilon}\right)^{m/2q} du.$$

Appealing to Hölder's inequality

$$(u+v)^p \leqslant 2^{p-1}(u^p + v^p), \qquad u, v \geqslant 0, \ p \geqslant 1, \tag{2.10}$$

we find that

$$|D_x^m G_q(x; t)| \leqslant \frac{e^{tA_{q,\gamma} v^{2q} - vx}}{\pi} \left(\frac{m}{2qe\varepsilon}\right)^{m/2q} \times \int_0^{\infty} e^{-t\gamma u^{2q} + \varepsilon 2^{q-1}(u^{2q}+v^{2q})} \, du.$$

Choosing

$$\varepsilon = \frac{t\gamma}{2^{q-1}} \delta,$$

where $0 < \delta < 1$, we have

$$|D_x^m G_q(x; t)| \leqslant \frac{e^{tA_{q,\gamma} v^{2q} - vx + t\gamma \delta v^{2q}}}{\pi} \times \left(\frac{2^{q-1} m}{2qe\gamma \delta t}\right)^{m/2q} \int_0^{\infty} e^{-t\gamma u^{2q}(1-\delta)} \, du.$$

If we now let

$$v = \frac{|x|}{x} \left(\frac{|x|}{2qt(A_{q,\gamma} + \gamma\delta)}\right)^{1/(2q-1)},$$

we note that

$$|D_x^m G_q(x;t)| \leqslant \exp\left[-\frac{2q-1}{2q} \left[\frac{x^{2q}}{2qt(A_{q,\gamma}+\gamma\delta)} \right]^{1/(2q-1)} \right]$$
$$\times \frac{\Gamma(1/2q)}{2q\pi[t\gamma(1-\delta)]^{1/2q}} \left(\frac{2^{q-1}m}{2qe\gamma\,\delta t} \right)^{m/2q}.$$

We have thus established the following result.

THEOREM 2.2. *For $t>0$, $0<\gamma, \delta<1$,*

$$|G_q(x,t)| \leqslant \frac{\Gamma(1/2q)}{2q\pi} (t\gamma)^{-1/2q} e^{-C_{q,\gamma}(x^{2q}/t)^{1/(2q-1)}}, \tag{2.11}$$

and, for $m=1, 2, 3, \ldots$,

$$|D_x^m G_q(x,t)| \leqslant \frac{\Gamma(1/2q)}{2q\pi[\gamma(1-\delta)]^{1/2q}} \left(\frac{2^q m}{4qe\gamma\delta} \right)^{m/2q}$$
$$\times t^{-(m+1)/2q} e^{-C_{q,\gamma,\delta}(x^{2q}/t)^{1/2q-1)}}, \tag{2.12}$$

where

$$C_{q,\gamma,\delta} = \frac{2q-1}{2q} (2q[A_{q,\gamma}+\gamma\delta])^{-1/(2q-1)} \tag{2.13}$$

and

$$C_{q,\gamma} = C_{q,\gamma,0}. \tag{2.14}$$

As an immediate consequence, we have the following.

COROLLARY 2.3. *For $t>0$, $m=0, 1, 2, \ldots$,*

$$\lim_{x \to \pm\infty} D_x^m G_q(x;t) = 0. \tag{2.15}$$

3. THE HEAT POLYNOMIALS

The heat polynomials are polynomial solutions of (2.1), valid for all values of the variables x, t, and given by

$$p_{n,q}(x,t) = n! \sum_{k=0}^{[n/2q]} (-1)^{(q+1)k} \frac{x^{n-2qk}}{(n-2qk)!} \frac{t^k}{k!}, \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{3.1}$$

Straightforward calculations result, for these polynomials, in the hypergeometric series representations

$$p_{n,q}(x,t)=x^n\,{}_{2q}F_0\left(\frac{-n}{2q},\ldots,\frac{2q-1-n}{2q};\,-;\,(-1)^{q+1}\left(\frac{2q}{x}\right)^{2q}t\right), \tag{3.2}$$

and, for $n=2qN+s$, $N\in\mathbb{N}_0$, $s=0,1,\ldots,2q-1$,

$$\begin{aligned} p_{2qN+s,q}(x,t) &= x^s((-1)^{q+1}t)^N\frac{(s+1)_{2qN}}{N!} \\ &\quad\times{}_1F_{2q-1}\left(-N;\frac{s+1}{2q},\ldots,\frac{2q-1}{2q},\frac{2q+1}{2q},\ldots,\frac{s+2q}{2q};\frac{(-1)^q}{t}\left(\frac{x}{2q}\right)^{2q}\right), \end{aligned} \tag{3.3}$$

where, in the last equation, we used the Gauss–Legendre multiplication formula [3].

We note the special cases

$$p_{n,q}(x,0)=x^n \tag{3.4}$$

and

$$p_{2qN+s,q}(0,t)=(-1)^{(q+1)N}\frac{(2qN)!}{N!}t^N\delta_{s,0}, \tag{3.5}$$

where $\delta_{i,j}$ is the Kronecker delta.

The exponential generating function for the heat polynomials given by

$$e^{xz+(-1)^{q+1}z^{2q}t}=\sum_{n=0}^{\infty}p_{n,q}(x,t)\frac{z^n}{n!} \tag{3.6}$$

can be readily established by multiplying the series for e^{xz} and for $e^{(-1)^{q+1}z^{2q}t}$ to obtain a simple series in z and taking note of (3.1).

The degree of the derivatives of $p_{n,q}(x,t)$ is reduced by differentiation with respect to eeither variable as follows:

$$\frac{\partial}{\partial x}p_{n,q}(x,t)=np_{n-1,q}(x,t) \tag{3.7}$$

and

$$\frac{\partial}{\partial t}p_{n,q}(x,t)=(-1)^{q+1}\frac{n!}{(n-2q)!}p_{n-2q,q}(x,t). \tag{3.8}$$

Standard calculations establish the fact that the polynomials satisfy the linear homogeneous differential equation of order $2q$ in the variable x, with t fixed,

$$2qt(-1)^{q+1}\frac{d^{2q}}{dx^{2q}}p_{n,q}(x,t)+x\frac{d}{dx}p_{n,q}(x,t)-np_{n,q}(x,t)=0, \tag{3.9}$$

and the recursion relation

$$p_{n+1,q}(x,t) - x p_{n,q}(x,t) + (-1)^q 2qt \frac{n!}{(n-2q+1)!} p_{n-2q+1,q}(x,t) = 0. \tag{3.10}$$

To obtain the Poisson integral transform representation $\int_{-\infty}^{\infty} G_q(x-y;t)\, y^n\, dy$ for the heat polynomials, we first need to establish the following important moment property for the source solution.

LEMMA 3.1. *For $t > 0$, $m \in \mathbb{N}_0$,*

$$I_{m,q}(t) = \int_{-\infty}^{\infty} x^m G_q(x;t)\, dx$$

$$= \frac{(2qn)!}{n!} ((-1)^{q+1} t)^n \delta_{m,2qn}, \qquad n = 0, 1, 2, \ldots. \tag{3.11}$$

Proof. Since $G_q(x,t)$ is an even function of x, it is clear that if m is odd, the integral vanishes.

Now, for $m = 2k$, $\lambda > 0$, consider

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} I_{2k,q}(t) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int_{-\infty}^{\infty} x^{2k} G_q(x,t)\, dx \\
&= \int_{-\infty}^{\infty} e^{-\lambda x^2} G_q(x;t)\, dx \\
&= \frac{1}{\pi} \int_0^{\infty} e^{-ty^{2q}}\, dy \int_{-\infty}^{\infty} e^{-\lambda x^2} \cos xy\, dx \\
&= \int_{-\infty}^{\infty} e^{-ty^{2q}}\, dy \left(\frac{1}{\pi} \int_0^{\infty} e^{-\lambda x^2} \cos xy\, dx \right) \\
&= \int_{-\infty}^{\infty} G_1(y,\lambda) \sum_{n=0}^{\infty} \frac{(-ty^{2q})^n}{n!}\, dy \\
&= \frac{1}{\sqrt{4\pi\lambda}} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} 2 \int_0^{\infty} e^{-y^2/4\lambda}\, y^{2qn}\, dy \\
&= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-2^{2q}\lambda^q t)^n}{n!} \Gamma\left(\frac{2qn+1}{2} \right) \\
&= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \frac{(2qn)!}{(qn)!} \lambda^{qn}.
\end{aligned}$$

Equating coefficients of powers of λ, we obtain (3.11). ∎

We now can express the heat polynomials as a convolution transform of monomials, with the translated source solution as kernel.

THEOREM 3.2. *For $t>0$ and any real x,*

$$p_{n,q}(x,t)=\int_{-\infty}^{\infty} y^n G_q(x-y;t)\,dy. \tag{3.12}$$

Proof. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} y^n G_q(x-y;t)\,dy\\
&\quad=\int_{-\infty}^{\infty} G_q(y;t)(x-y)^n\,dy\\
&\quad=\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}(-1)^k x^{n-k}\int_{-\infty}^{\infty} y^k G_q(y;t)\,dy.
\end{aligned}$$

An appeal to the preceding lemma yields

$$\begin{aligned}
&\int_{-\infty}^{\infty} y^n G_q(x-y;t)\,dy\\
&\quad=\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}(-1)^k x^{n-k}\frac{(2ql)!}{l!}((-1)^{q+1}t)^l\,\delta_{k,2ql}\\
&\quad=n!\sum_{l=0}^{[n/2q]}(-1)^{(q+1)l}\frac{t^l}{l!}\frac{x^{n-2ql}}{(n-2ql)!}\\
&\quad=p_{n,q}(x,t). \quad \blacksquare
\end{aligned}$$

The result is also immediate by operational calculus. Indeed, we have, by (3.11),

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-sy}G_q(y;t)\,dy&=\sum_{m=0}^{\infty}\frac{s^{2qm}}{(2qm)!}\frac{(2qm)!}{m!}((-1)^{(q+1)}t)^m\\
&=e^{(-1)^{q+1}ts^{2q}}.
\end{aligned} \tag{3.13}$$

If we thus replace the variable s by the operator D_x, we have, for a suitable function f,

$$\int_{-\infty}^{\infty} G_q(y,t)\,e^{-yD_x}f(x)\,dy=e^{(-1)^{q+1}t\,D_x^{2q}}f(x)$$

or

$$\int_{-\infty}^{\infty} G_q(y;t)\, f(x-y)\, dy = \sum_{m=0}^{\infty} (-1)^{(q+1)m} \frac{t^m}{m!} f^{(2qm)}(x).$$

It follows that if $f(x) = x^n$, we have (3.12).

In addition to (3.12), we have a complex integral representation for the heat polynomials.

THEOREM 3.3. *For $t > 0$ and any real x,*

$$p_{n,q}(x, (-1)^q t) = \int_{-\infty}^{\infty} G_q(y+ix, t)(iy)^n\, dy. \tag{3.14}$$

Proof. By Cauchy's theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} G_q(y+ix; t)(iy)^n\, dy \\
&\quad = \int_{-\infty}^{\infty} G_q(y, t)(x+iy)^n\, dy \\
&\quad = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} x^{n-k} i^k \int_{-\infty}^{\infty} G_q(y, t)\, y^k\, dy,
\end{aligned}$$

or, invoking (3.11), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} G_q(y+ix; t)(iy)^n\, dy \\
&\quad = \sum_{k=0}^{[n/2q]} \frac{n!}{(2qk)!\,(n-2qk)!} x^{n-2qk} (-1)^{qk} \frac{(2qk)!}{k!} ((-1)^{q+1} t)^k \\
&\quad = \sum_{k=0}^{[n/2q]} \frac{n!\, x^{n-2qk}}{(n-2qk)!} \frac{((-1)^q t)^k}{k!} (-1)^{(q+1)k} \\
&\quad = p_{n,q}(x, (-1)^q t). \quad \blacksquare
\end{aligned}$$

4. The Associated Functions

The functions $w_{n,q}(x, t)$ associated with the heat polynomials are given by

$$w_{n,q}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ty^{2q} + ixy} (-2iy)^n\, dy. \tag{4.1}$$

We note that $w_{0,q}(x, t) = G_q(x; t)$, and, indeed,

$$w_{n,q}(x, t) = (-2)^n D_x^n \left(\frac{1}{\pi} \int_0^\infty e^{-ty^{2q}} \cos xy \, dy \right)$$
$$= (-2)^n D_x^n G_q(x; t). \tag{4.2}$$

A series representation for $w_{n,q}(x, t)$ may be derived from (4.1). We have

$$w_{n,q}(x, t) = \frac{(-2i)^n}{2\pi} \int_{-\infty}^{\infty} e^{-ty^{2q}} y^n \, dy \sum_{k=0}^{\infty} \frac{(ixy)^k}{k!}$$
$$= \frac{(-2i)^n}{2\pi} \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} (1 + (-1)^{k+n}) \int_0^\infty e^{-ty^{2q}} y^{n+k} \, dy$$
$$= \frac{(-2i)^n}{2q\pi} \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \frac{1 + (-1)^{k+n}}{2} \frac{\Gamma((k+n+1)/2q)}{t^{(k+n+1)/2q}}, \tag{4.3}$$

where termwise integration is justified by the absolute convergence of the final series.

Considering separately even and odd indices n, we may represent the $w_{n,q}(x, t)$ in terms of hypergeometric series. Calculations analogous to those in the derivation of (2.4) give us

$$w_{2n,q}(x, t)$$
$$= \frac{(-1)^n}{2q\pi} 2^{2n} t^{-(2n+1)/2q} \sum_{l=0}^{q-1} \left(\frac{-x^2}{t^{1/q}} \right)^l \frac{\Gamma((2n+2l+1)/2q)}{(2l)!}$$
$$\times {}_1F_{2q-1}\left(\frac{2n+2l+1}{2q}; \frac{2l+1}{2q}, ..., \frac{2q-1}{2q}, \frac{2q+1}{2q}, ..., \frac{2l+2q}{2q}; \frac{(-1^q(x/2q)^{2q}}{t} \right) \tag{4.4}$$

and

$$w_{2n+1,q}(x, t)$$
$$= \frac{(-1)^n 2^{2n+1}}{2q\pi} t^{-(2n+3)/(2q)} \sum_{l=0}^{q-1} \frac{(-1)^l x^{2l+1}}{t^{l/q}} \frac{\Gamma((2n+2l+3)/2q)}{(2l+1)!}$$
$$\times {}_1F_{2q-1}\left(\frac{2n+2l+3}{2q}; \frac{2l+2}{2q}, ..., \frac{2q-1}{2q}, \frac{2q+1}{2q}, ..., \frac{2l+2q+1}{2q}; \frac{(-1)^q(x/2q)^{2q}}{t} \right). \tag{4.5}$$

The translated source solution is a generating function for the associated heat polynomials, and we have

$$G_q(z-x; t) = \sum_{n=0}^{\infty} w_{n,q}(x, t) \frac{z^n}{2^n n!}, \tag{4.6}$$

an equation that is readily established by introducing (4.3) and obtaining on the right the series (2.3) for $G_q(z-x; t)$.

By using, in turn, generating functions and integral representations for $p_{n,q}(x, t)$ and $w_{n,q}(x, t)$, we can derive an addition property for the source solution. Consider, for $s, t > 0$,

$$\sum_{n=0}^{\infty} \frac{p_{n,q}(x, t)\, w_{n,q}(y, s)}{2^n n!}.$$

Appealing to (3.12) and (4.6), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{p_{n,q}(x, t)\, w_{n,q}(y, s)}{2^n n!} \\ &= \int_{-\infty}^{\infty} G_q(x-z; t)\, dz \sum_{n=0}^{\infty} \frac{z^n w_{n,q}(y, s)}{2^n n!} \\ &= \int_{-\infty}^{\infty} G_q(x-z; t)\, G_q(z-y; s)\, dz, \end{aligned} \tag{4.7}$$

where the interchange of summation and integration can be justified by an appeal to (2.11), (2.12), and (4.2).

On the other hand, using (4.1) and (3.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} &\frac{p_{n,q}(x, t)\, w_{n,q}(y, s)}{2^n n!} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{izy - sz^{2q}}\, dz \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} p_{n,q}(x, t) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{izy - zsz^{2q} - izx + (-1)^{q+1}(-iz)^{2q}t}\, dz \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(s+t)z^{2q} - iz(x-y)}\, dz \\ &= G_q(x-y; s+t). \end{aligned} \tag{4.8}$$

We thus have the addition formula for the source solution

$$\int_{-\infty}^{\infty} G_q(x-z; t)\, G_q(z-y; s)\, dz = G_q(x-y; s+t). \tag{4.9}$$

Further, using (4.6), with z complex, and (3.14), we have

$$\int_{-\infty}^{\infty} G_q(z+ix; t)\, G_q(y-iz; s)\, dz = G_q(x-y; s+(-1)^q t). \tag{4.10}$$

The associated functions $w_{n,q}(x, t)$ are solutions of the differential equation

$$(-1)^q 2qt\, D_x^{2q} w_{n,q}(x, t) - x\, D_x w_{n,q}(x, t) - (n+1)\, w_{n,q}(x, t) = 0, \qquad (4.11)$$

and satisfy the recurrence relation

$$(x/2)\, w_{n+2-2q,q}(x, t) - (n+2-2q)\, w_{n+1-2q,q}(x, t) + (-1)^q 2^{1-2q} qt w_{n+1,q}(x, t) = 0. \qquad (4.12)$$

5. Generalized Hermite Polynomials and Their Associated Functions

The classical heat polynomials $p_n(x, t) = p_{n,1}(x, t)$ are related to Hermite polynomials of degree n by the equation

$$p_n(x, t) = (-t)^{n/2} H_n\left(\frac{x}{(-4t)^{1/2}}\right).$$

Defining the generalized Hermite polynomials $H_{n,q}(x)$ by

$$H_{n,q}(x) = n! \sum_{k=0}^{[n/2q]} \frac{(-1)^{qk}(2^{1/q}x)^{n-2qk}}{k!\,(n-2qk)!}$$
$$= (2^{1/q}x)^n\, {}_{2q}F_0\left(\frac{-n}{2q}, \ldots, \frac{-n+2q-1}{2q}; -; \frac{(-1)^q}{4}\left(\frac{2q}{x}\right)^{2q}\right), \qquad (5.1)$$

we have that

$$p_{n,q}(x, t) = (-t)^{n/2q} H_{n,q}\left(\frac{x}{(-4t)^{1/(2q)}}\right), \qquad (5.2)$$

where the generalized Hermite polynomials satisfy the differential equation

$$(-1)^{q+1} q H_{n,q}^{(2q)}(x) - 2x H'_{n,q}(x) + 2n H_{n,q}(x) = 0 \qquad (5.3)$$

and the recurrence relation

$$H_{n+1,q}(x) - 2^{1/q} x H_{n,q}(x) + (-1)^{q+1} 2q \frac{n!}{(n+1-2q)!} H_{n+1-2q,q}(x) = 0. \qquad (5.4)$$

We note that, as a consequence of (5.2) and (3.3), if $n = 2qN + s$, $s = 0, 1, \ldots, 2q-1$, the generalized Hermite polynomials are also given by

$$H_{2qN+s}(x) = (-1)^{qN} \frac{(s+1)_{2qN}}{N!} (2^{1/q}x)^s$$
$$\times {}_1F_{2q-1}\left(-N; \frac{s+1}{2q}, ..., \frac{2q-1}{2q}, \frac{2q+1}{2q}, ..., \frac{s+2q}{2q}; (-1)^{q+1} 4\left(\frac{x}{2q}\right)^{2q}\right). \tag{5.5}$$

A generating function for these polynomials is

$$\sum_{n=0}^{\infty} H_{n,q}(x) \frac{z^n}{n!} = e^{2^{1/q}xz + (-1)^q z^{2q}}. \tag{5.6}$$

It follows that

$$\sum_{n=0}^{\infty} H_{n,q}(x) \frac{(2^{(q-1)/q}z)^n}{n!}$$
$$= e^{2xz + (-1)^q 2^{2q-2} z^{2q}}$$
$$= e^{2xz - z^2} e^{z^2(1 + (-1)^q 2^{2q-2} z^{2q-2})}$$
$$= \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{z^{2m}}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^{qk} (2z)^{(2q-2)k}$$
$$= \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^{qk}(2z)^{(2q-2)k}}{k!} \sum_{m=k}^{\infty} \frac{z^m}{(m-k)!}$$
$$= \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^{qk}(2z)^{(2q-2)k}}{k!} \sum_{m=qk}^{\infty} \frac{z^{2(m-2k+k)}}{(m-qk)!}$$
$$= \sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{z^{2m}}{m!} \sum_{k=0}^{[m/q]} (-1)^{qk} 2^{2(q-1)k} \frac{m!}{k!\,(m-qk)!}$$
$$= \sum_{m=0}^{\infty} \frac{z^{2m}}{m!} \sum_{k=0}^{[m/q]} (-1)^{qk} 2^{2(q-1)k} \frac{m!}{k!\,(m-qk)!}$$
$$\times \sum_{n=2m}^{\infty} H_{n-2m}(x) \frac{z^{n-2m}}{(n-2m)!}$$
$$= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{[n/2]} \frac{H_{n-2m}(x)}{(n-2m)!} \sum_{k=0}^{[m/q]} (-1)^{qk} 2^{2(q-1)k} \frac{1}{k!\,(m-qk)!}.$$

Equating coefficients of z^n, we find that the generalized Hermite polynomials can be expressed as a linear combination of Hermite polynomials by

$$H_{n,q}(x) = 2^{((1-q)/q)n} n! \sum_{m=0}^{[n/2]} a_{m,q} \frac{H_{n-2m}(x)}{(n-2m)!}, \tag{5.7}$$

where

$$a_{m,q}=\sum_{k=0}^{[m/q]}\frac{(-1)^{qk}2^{2(q-1)k}}{k!\,(m-qk)!}$$
$$=\sum_{k=0}^{[m/q]}\frac{2^{2(q-1)k}(-m)_{qk}}{k!\,m!}$$
$$=\frac{1}{m!}\,{}_qF_0\left(\frac{-m}{q},\frac{1-m}{q},\ldots,\frac{q-1-m}{q};-;q^q2^{2q-2}\right). \tag{5.8}$$

Appealing to (5.2), we have a corresponding expansion of the heat polynomials $p_{n,q}(x,t)$ in terms of the classical polynomials $p_n(x,t)$ given by

$$p_{n,q}\left(x,\frac{-t^q}{4}\right)=\sum_{m=0}^{[n/2]}a_{m,q}\frac{n!}{(n-2m)!}\left(\frac{t}{4}\right)^m p_{n-2m}\left(x,\frac{-t}{4}\right). \tag{5.9}$$

Corresponding to the associated functions $w_{n,q}(x,t)$, we define the functions $W_{n,q}(x)$ by

$$w_{n,q}(x,t)=t^{-(n+1)/2q}W_{n,q}\left(\frac{x}{(4t)^{1/(2q)}}\right). \tag{5.10}$$

Substituting (5.10) into (4.11), we find that

$$(-1)^{q+1}qW_{n,q}^{(2q)}(x)+(2xW_{n,q}(x))'+2nW_{n,q}(x)=0, \tag{5.11}$$

so that the differential equation satisfied by $H_{n,q}(x)$ and $W_{n,q}(x)$ are adjoint to each other.

6. Homogeneity

A function $u(x,t)\in H$ is said to be homogeneous of degree n if

$$u(\lambda x,\lambda^{2q}t)=\lambda^n u(x,t), \tag{6.1}$$

for any positive number λ.

From the series (2.3), (3.1), and (4.3), respectively, we can readily conclude that

$$G_q(x,t) \text{ is homogeneous of degree } -1; \tag{6.2}$$

$$p_{n,q}(x;t) \text{ is homogeneous of degree } n; \tag{6.3}$$

$$w_{n,q}(x,t) \text{ is homogeneous of degree } -n-1. \tag{6.4}$$

Further, all temperatures $u(x, t)$ homogeneous of degree r satisfy Eqs. (1.1) and (6.1) with $n = r$, and thus are also solutions of the differential equation

$$2qt(-1)^{q+1} D_x^{2q} u(x, t) + x D_x u(x, t) - ru(x, t) = 0. \tag{6.5}$$

If we let

$$u(x, t) = \sum_{k=0}^{\infty} a_{k,q}(t) \frac{x^k}{k!},$$

we find that the coefficients must satisfy the equation

$$a_{k+2qm,q}(t) = (-1)^{qm} \left(\frac{k-r}{2q}\right)_m t^{((k-r)/2q)-m} a_{k,q}(1), \qquad k = 0, 1, ..., 2q-1.$$

It follows that

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{2q-1} \sum_{m=0}^{\infty} a_{k+2qm,q}(t) \frac{x^{k+2qm}}{(k+2qm)!} \\ &= \sum_{k=0}^{2q-1} a_{k,q}(1)\, {}_r u_{k,q}(x, t), \end{aligned}$$

where

$$\begin{aligned} &{}_r u_{k,q}(x, t) \\ &\quad = \frac{x^k}{k!} t^{(r-k)/2q} \sum_{m=0}^{\infty} \frac{((k-r)/2q)_m}{(k+1)_{2qm}} \left(\frac{(-1)^q x^{2q}}{t}\right)^m, \qquad k = 0, 1, ..., 2q-1, \\ &\quad = \frac{x^k}{k!} t^{(r-k)/2q} \\ &\qquad \times {}_1F_{2q-1}\left(\frac{k-r}{2q}; \frac{k+1}{2q}, ..., \frac{2q-1}{2q}, \frac{2q+1}{2q}, ..., \frac{k+2q}{2q}; \frac{(-1)^q}{t} \left(\frac{x}{2q}\right)^{2q}\right) \end{aligned} \tag{6.6}$$

denote $2q$ linearly independent temperatures which are homogeneous of degree r.

Now, for $r = 2qN + s$, $N \in \mathbb{N}_0$, $s = 0, 1, ..., 2q-1$,

$$\begin{aligned} &{}_r u_{s,q}(x, t) \\ &= \frac{x^s}{s!} t^N \\ &\quad \times {}_1F_{2q-1}\left(-N; \frac{r+1}{2q} - N, ..., \frac{2q-1}{2q}, \frac{2q+1}{2q}, ..., \frac{r+2q}{2q} - N; \frac{(-1)^q}{t} \left(\frac{x}{2q}\right)^{2q}\right) \end{aligned}$$

so that, comparing with (3.3), we have

$$ {}_r u_{s,q}(x,t) = (-1)^{(q+1)N} \frac{N!}{r!} p_{r,q}(x,t), \qquad N = \left[\frac{r}{2q}\right]. \tag{6.7}$$

Hence we have established that, for any non-negative integer r, there is an integer s, $s = 0, 1, ..., 2q-1$, such that ${}_r u_{s,q}(x,t)$ has a terminating hypergeometric expansion which is some multiple of $p_{r,q}(x,t)$, as noted in (6.7).

Now, if $r = -1$, we note from the second equation of (6.6) that for $k = 0, 1, ..., 2q-2$,

$$ {}_{-1}u_{k,q}(x,t) = \frac{x^k}{k!} t^{-(k+1)/2q} $$
$$ \times {}_0F_{2q-2}\left(-; \frac{k+2}{2q}, ..., \frac{2q-1}{2q}, \frac{2q+1}{2q}, ..., \frac{k+2q}{2q}; \frac{(-1)^q}{t}\left(\frac{x}{2q}\right)^{2q}\right). $$

Referring to (2.4), we thus have that

$$ G_q(x,t) = \frac{1}{2q\pi} \sum_{l=0}^{q-1} (-1)^l \Gamma\left(\frac{2l+1}{2q}\right) {}_{-1}u_{2l,q}(x,t). \tag{6.8}$$

If $r = -2m-1$, $m = 0, 1, ...,$ then it follows readily from the second equation of (6.6) and (4.4) that

$$ w_{2m,q}(x,t) = \frac{(-1)^m 2^{2m}}{2q\pi} \sum_{l=0}^{q-1} (-1)^l \Gamma\left(\frac{2m+2l+1}{2q}\right) {}_{-2m-1}u_{2l,q}(x,t), \tag{6.9}$$

whereas if $r = -2m-2$, we have from (4.5),

$$ w_{2m+1,q}(x,t) $$
$$ = \frac{(-1)^m 2^{2m+1}}{2q\pi} \sum_{l=0}^{q-1} (-1)^l \Gamma\left(\frac{2m+2l+3}{2q}\right) {}_{-2m-2}u_{2l+1,q}(x,t). \tag{6.10}$$

The properties we developed in this paper for the source solution, which has a central role in the theory, the heat polynomials, and their associated functions will form the basis for our establishing sufficient conditions for the series representation of a solution of (1.1).

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